

# Effects of turbulent mixing on the nonequilibrium critical behaviour

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**Abstract.** We study effects of turbulent mixing on the critical behaviour of a nonequilibrium system near its second-order phase transition between the absorbing and fluctuating states. The model describes the spreading of an agent (e.g., infectious disease) in a reaction-diffusion system and belongs to the universality class of the directed bond percolation process, also known as simple epidemic process, and is equivalent to the Reggeon field theory. The turbulent advecting velocity field is modelled by the Obukhov–Kraichnan’s rapid-change ensemble: Gaussian statistics with the correlation function  $\langle vv \rangle \propto \delta(t - t') k^{-d-\xi}$ , where  $k$  is the wave number and  $0 < \xi < 2$  is a free parameter. Using the field theoretic renormalization group we show that, depending on the relation between the exponent  $\xi$  and the spatial dimension  $d$ , the system reveals different types of large-scale asymptotic behaviour, associated with four possible fixed points of the renormalization group equations. In addition to known regimes (ordinary diffusion, ordinary directed percolation process, and passively advected scalar field), existence of a new nonequilibrium universality class is established, and the corresponding critical dimensions are calculated to first order of the double expansion in  $\xi$  and  $\varepsilon = 4 - d$  (one-loop approximation). It turns out, however, that the most realistic values  $\xi = 4/3$  (Kolmogorov’s fully developed turbulence) and  $d = 2$  or  $3$  correspond to the case of passive scalar field, when the nonlinearity of the Reggeon model is irrelevant and the spreading of the agent is completely determined by the turbulent transfer.

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## 1. Introduction

Over the past few decades, constant interest has been attracted by the spreading processes and corresponding nonequilibrium phase transitions; see Refs. [1]–[10] and the literature cited therein. Spreading processes are ubiquitous in the Nature and are encountered in physical, chemical, biological, ecological and sociological systems: autocatalytic reactions, percolation in porous media, forest fires, epidemic diseases, and so on. For definiteness, in the following we will use the terminology of the latter case (spreading of a disease).

Depending on the conditions, the spreading of an agent (disease) can either continue over the whole population, or terminate after some time. In the first case, the system evolves to a stationary (but not thermally equilibrium) active state, in which the sick and healthy individuals coexist, passing repeatedly through infection and healing events, and their densities are fluctuating (random) quantities. In the second case, when the probability of being healed is large enough in comparison with the probability to be infected, the system is “trapped” in an absorbing (inactive) state where all the individuals are healthy and all the fluctuations cease completely. The transitions between these fluctuating and absorbing phases are continuous; they are especially interesting as examples of nonequilibrium critical behaviour.

By analogy with the equilibrium second-order transitions [11, 12] it is expected that, near the critical point, many details of a specific spreading process become irrelevant, so that the critical behaviour of different systems appears identical and can be described by a certain *universality class*. The aim of the theory is to identify possible universality classes and to calculate their universal characteristics (such as critical exponents, scaling functions) on the basis of an appropriate theoretic model and within a controlled approximation or a regular perturbation scheme.

As a rule, the spreading phenomena are modelled by various stochastic reaction-diffusion processes on a lattice; see Ref. [1] for a detailed discussion. In the continuum limit, they can be mapped onto certain field theoretic models with the aid of special techniques [13, 14]. Then the powerful tools [11, 12] of the field theoretic renormalization group (RG) can be applied to the investigation of their critical behaviour.

The most typical processes belong to the universality class of the so-called *directed bond percolation process* [1]–[3], which in the field theoretic formulation is equivalent to the well-known Reggeon field theory [15]. It was conjectured in [4] that the critical behaviour always belongs to this “DP class” provided the absorbing state of the process is unique, the order parameter is one-dimensional, there are no specific symmetries, no coupling with additional “slow” degrees of freedom and no long-range interactions. Thus the DP process is expected [4] to play in the theory of nonequilibrium phase transitions the same paradigmatic role as the standard  $\lambda\phi^4$  model [11, 12] does in the theory of equilibrium critical behaviour. It is also sometimes referred to as simple epidemic process with recovery or as Gribov’s process; the stochastic version of the Schlögl’s first reaction also belongs to this class [1]–[3]. The corresponding critical

behaviour is rather well understood: above the upper critical dimension  $d_c = 4$  the critical exponents are given by the mean-field theory, and below  $d_c$  they have been calculated to second order of the expansion in the deviation  $\varepsilon = d_c - d$  of the spatial dimension  $d$  from the upper critical value; see the discussion and references in Ref. [3]. Recently, the first experimental observation of this universality class was achieved in the transition between two topologically different turbulent states of electrohydrodynamic convection of a nematic liquid crystal [5], with a firm agreement between theoretical and experimental values of all the critical exponents.

Extensive numerical and analytical investigations of various models of the spreading processes have shown that effects of immunization [1]–[3], [7], long-range interactions [6]–[8], presence of quenched (static) disorder [9] or critical fluctuations [10] in the medium where the spreading occurs can lead to appearance of completely new universality classes, with different values of the upper critical dimensions  $d_c$ , different sets of critical exponents and interesting crossovers between various critical regimes.

In this paper we will consider the spreading of a nonconserved agent in a turbulent medium and study the effects of turbulent stirring and mixing on the critical behaviour near the phase transition between the absorbing and the fluctuating phases. Turbulence is hardly avoidable in the chemical catalytic reactions and can play decisive role if a disease is transferred by flying insects or birds.

Much work has been devoted to the investigation of the effects of various kinds of imposed deterministic or chaotic flows (laminar shear flows, turbulent convection and so on) on the behaviour of the critical or nearly critical fluids, like binary liquid mixtures near and below their consolution points; see e.g. the papers [16]–[25] and references therein. It was shown that the flow can destroy the usual critical behaviour, typical of the  $\lambda\phi^4$  model, which changes to the mean-field behaviour [16, 17] or to a complex behaviour described by new nonequilibrium universality classes [19, 20, 21]. This problem is closely related to another interesting issue: the effects of imposed flows on the dynamics of phase ordering – the growth of order through domain coarsening (spinodal decomposition) and formation of various nontrivial spatiotemporal patterns; see [22]–[25].

Theoretical description of fully developed turbulence on the basis of a microscopic dynamic model (like e.g. the stirred Navier-Stokes equation) remains essentially an open problem [26]. In this paper, we will describe the turbulent mixing by a Gaussian velocity ensemble with prescribed statistics, which dates back to Obukhov, Batchelor, Kazantsev and Kraichnan and now is widely known as the rapid-change model [27]. In spite of their relative simplicity, the models of passive (no feedback on the velocity) scalar fields advected by such “synthetic” velocity ensembles have attracted serious attention recently because of the deep insight they offer into the origin of intermittency and anomalous (multi)scaling of the genuine turbulent heat or mass transport; see the review paper [27] and references therein. The RG approach to that problem is reviewed in Ref. [28].

In the rapid-change model, the pair velocity correlation function is taken in the

form  $\langle vv \rangle \propto \delta(t - t') k^{-d-\xi}$ , where  $k$  is the wave number and  $0 < \xi < 2$  a free parameter with the most realistic (Kolmogorov) value  $\xi = 4/3$ . Vanishing of the correlation time ensures the Galilean symmetry of the problem, while a power-law dependence on the wave number  $k$  mimics the real scaling properties of fully developed turbulence.

The plan of the paper is as follows. In section 2 we present the detailed description of the model and its field theoretic formulation. In section 3 we analyze the ultraviolet (UV) divergences of the model, relaying upon the canonical dimensions and additional symmetry considerations. We show that the model is multiplicatively renormalizable and write the renormalized action functional.

In section 4 we derive the RG equations and introduce the RG functions ( $\beta$  functions and anomalous dimensions  $\gamma$ ). In section 5 we identify four possible infrared (IR) attractive fixed points of the RG equations, and identify their ranges of stability in the  $\varepsilon$ - $\xi$  plane.

These fixed points correspond to four possible critical regimes of the model, with different sets of critical dimensions, as discussed in section 6. Three of them correspond to already known regimes: free (Gaussian) field theory, linear passive scalar advection by the Obukhov–Kraichnan ensemble (the nonlinearity in the agent’s density appears irrelevant), and the ordinary Gribov process (mixing by the turbulent field is irrelevant). The fourth fixed point corresponds to a new nonequilibrium universality class, with the new set of critical exponents that depend on the both parameters  $\varepsilon$  and  $\xi$  and can be systematically calculated as double series in these parameters.

The practical calculation of the renormalization constants, RG functions, regions of stability and critical dimensions is accomplished to the leading-order (one-loop) approximation; some of the results, however, are exact (valid to all orders of the double  $\varepsilon$ - $\xi$  expansion).

In section 6 we also consider, as a special consequence of the general scaling relations, the temporal evolution of a cloud of the advected agent’s particles (infected individuals), which differs, in general, from the well-known “1/2 law” for ordinary diffusion. Section 7 is reserved for discussion and conclusion.

## 2. Description of the model. Field theoretic formulation

In the Lagrangian formulation, spreading of an agent is described by a continuous stochastic diffusion-reaction equation of the form

$$\partial_t \psi(t, \mathbf{x}) = \lambda_0 \{ (-\tau_0 + \partial^2) \psi(t, \mathbf{x}) - g_0 \psi^2(t, \mathbf{x})/2 \} + \zeta(t, \mathbf{x}), \quad (2.1)$$

where  $\psi(t, \mathbf{x}) > 0$  is the agent’s density,  $\partial^2$  is the Laplace operator,  $\lambda_0$  and  $g_0$  are positive parameters,  $\tau_0$  — deviation of the infection probability from its critical value (analog of  $\tau_0 \propto (T - T_c)$  in equilibrium systems). The random Gaussian noise  $\zeta(t, \mathbf{x})$  with zero mean and a given correlation function

$$\langle \zeta(t, \mathbf{x}) \zeta(t', \mathbf{x}') \rangle = g_0 \lambda_0 \psi(t, \mathbf{x}) \delta(t - t') \delta^{(d)}(\mathbf{x} - \mathbf{x}') \quad (2.2)$$

mimics fluctuations in the system;  $d$  is the dimension of the  $\mathbf{x}$  space.

According to the general theorem [13] (see also the monographs [11, 12]), stochastic problem (2.1), (2.2) is equivalent to the field theoretic model of the doubled set of fields with action functional

$$\mathcal{S}(\psi, \psi^\dagger) = \psi^\dagger(-\partial_t + \lambda_0 \partial^2 - \lambda_0 \tau_0) \psi + \frac{g_0 \lambda_0}{2} ((\psi^\dagger)^2 \psi - \psi^\dagger \psi^2). \quad (2.3)$$

Here,  $\psi^\dagger = \psi^\dagger(t, \mathbf{x})$  is the auxiliary “response field” and the integration over the arguments of the fields is implied, for example

$$\psi^\dagger \partial_t \psi = \int dt \int d\mathbf{x} \psi^\dagger(t, \mathbf{x}) \partial_t \psi(t, \mathbf{x}).$$

This means that statistical averages of random quantities in the original stochastic problem (2.1), (2.2) can be represented as functional averages over the full set of fields with the weight  $\exp \mathcal{S}(\Phi)$ , and can therefore be viewed as the Green functions of the field theoretic model with the action (2.3). In particular, the linear response function of the problem (2.1), (2.2) is given by the Green function  $G = \langle \psi^\dagger \psi \rangle$  of the model (2.3).

More rigorous derivation of the field theory is based on the so-called master equation for the original reaction-diffusion process on the lattice, which involves a microscopic Hamiltonian written in terms of the creation-annihilation operators with further representation of the corresponding evolution operator by the coherent-state functional integral [14]. For the case at hand the resulting action functional coincides with (2.3) up to irrelevant terms; see also discussion of that approach in Refs. [2, 3].

One can argue that in the perturbation theory the condition  $\psi > 0$  can be neglected [4]; then the model (2.3) becomes equivalent to the Reggeon field theory [15] and acquires the symmetry with respect to the transformation

$$\psi(t, \mathbf{x}) \rightarrow \psi^\dagger(-t, -\mathbf{x}), \quad \psi^\dagger(t, \mathbf{x}) \rightarrow \psi(-t, -\mathbf{x}), \quad g_0 \rightarrow -g_0. \quad (2.4)$$

Reflection of the constant  $g_0$  is in fact unimportant because, as can easily be seen, the actual expansion parameter in the perturbation theory is  $g_0^2$  rather than  $g_0$  itself.

The model (2.3) corresponds to a standard Feynman diagrammatic technique with the only bare propagator  $G_0 = \langle \psi \psi^\dagger \rangle_0$  and the two triple vertices  $\sim (\psi^\dagger)^2 \psi, \psi^\dagger \psi^2$ . In the time-momentum and frequency-momentum representation  $G_0$  has the forms

$$G_0(t, k) = \theta(t) \exp \{ -\lambda_0(k^2 + \tau_0) \} \leftrightarrow G_0(\omega, k) = \frac{1}{-i\omega + \lambda_0(k^2 + \tau_0)}. \quad (2.5)$$

Here  $\theta(\dots)$  is the Heaviside step function, so that the propagator (2.5) is retarded. Then from the analysis of the diagrams one can check that the Green functions built solely from the field  $\psi$  or solely from  $\psi^\dagger$  necessarily contain closed circuits of retarded propagators (2.5) and therefore vanish identically. For the functions  $\langle \psi^\dagger \dots \psi^\dagger \rangle$  this fact is a general consequence of the causality, which is valid for any stochastic model; see e.g. the discussion in [12]. Then vanishing of the functions  $\langle \psi \dots \psi \rangle$  can be viewed as a consequence of the symmetry (2.4).

The stability in dynamical models like (2.3) implies that all the small fluctuations are damped out, so that the exact response function  $G = \langle \psi^\dagger \psi \rangle$  must decay for  $t \rightarrow \infty$ ; see e.g. sec. 5.5 in [12]. Then from the expression (2.5), which is the zero order

approximation for  $G$ , we conclude that the stability of the perturbative stationary state is lost for  $\tau_0 = 0$ , and for  $\tau_0 < 0$  the perturbations with smallest momenta grow in time. This growth is stabilized by the appearance of the nonzero constant mean  $\langle \psi \rangle$  and the higher-order correlation functions of the agent field  $\psi$ , so that the symmetry (2.4) is spontaneously broken. This is exactly the phase transition from the absorbing (normal) to the fluctuating (anomalous) states.

Coupling with the velocity field  $\mathbf{v} = \{v_i(t, \mathbf{x})\}$  is introduced by the replacement

$$\partial_t \rightarrow \nabla_t = \partial_t + v_i \partial_i, \quad (2.6)$$

in (2.1) and (2.3), where  $\partial_i = \partial/\partial x_i$  and  $\nabla_t$  is the Lagrangian (Galilean covariant) derivative. We will consider the case of incompressible flow, then the velocity field is divergence-free (transverse):  $\partial_i v_i = 0$ . In the real problem, the field  $\mathbf{v}(t, \mathbf{x})$  satisfies the Navier–Stokes equation. We will employ the rapid-change model, where the velocity obeys a Gaussian distribution with zero mean and correlation function

$$\langle v_i(t, \mathbf{x}) v_j(t', \mathbf{x}') \rangle = \delta(t - t') D_{ij}(\mathbf{r}), \quad \mathbf{r} = \mathbf{x} - \mathbf{x}' \quad (2.7)$$

$$D_{ij}(\mathbf{r}) = D_0 \int_{k>m} \frac{d\mathbf{k}}{(2\pi)^d} P_{ij}(\mathbf{k}) \frac{1}{k^{d+\xi}} \exp(i\mathbf{k}\mathbf{r}), \quad k \equiv |\mathbf{k}| \quad (2.8)$$

where  $P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j / k^2$  is the transverse projector,  $D_0 > 0$  is an amplitude factor and  $0 < \xi < 2$  is a free parameter with the most realistic (“Kolmogorov”) value  $\xi = 4/3$ . The cutoff in the integral (2.8) from below at  $k = m$ , where  $m \equiv 1/\mathcal{L}$  is the reciprocal of the integral turbulence scale  $\mathcal{L}$ , provides the IR regularization. Its precise form is unimportant; the sharp cutoff is the simplest choice for the practical calculations.

The full problem is equivalent to the field theoretic model of the three fields  $\Phi = \{\psi, \psi^\dagger, \mathbf{v}\}$  with the action functional

$$\mathcal{S}(\Phi) = \psi^\dagger (-\nabla_t + \lambda_0 \partial^2 - \lambda_0 \tau_0) \psi + \frac{\lambda_0 g_0}{2} ((\psi^\dagger)^2 \psi - \psi^\dagger \psi^2) + \mathcal{S}(\mathbf{v}), \quad (2.9)$$

which is obtained from (2.3) by the replacement (2.6) and adding the term corresponding to the Gaussian averaging over the field  $\mathbf{v}$  with the correlator (2.8):

$$\mathcal{S}(\mathbf{v}) = -\frac{1}{2} \int dt \int d\mathbf{x} \int d\mathbf{x}' v_i(t, \mathbf{x}) D_{ij}^{-1}(\mathbf{r}) v_j(t, \mathbf{x}'), \quad (2.10)$$

where

$$D_{ij}^{-1}(\mathbf{r}) \propto D_0^{-1} r^{-2d-\xi}$$

is the kernel of the inverse linear operation for the function  $D_{ij}(\mathbf{r})$  in (2.8).

In addition to (2.5), the Feynman diagrams for the model (2.9) involve the propagator  $\langle vv \rangle_0$  specified in (2.7), (2.8) and the new vertex  $-\psi^\dagger (v \partial) \psi$ .

The role of the coupling constants in the ordinary perturbation theory is played by the two parameters

$$u_0 = g_0^2 \sim \Lambda^{4-d}, \quad w_0 = D_0/\lambda_0 \sim \Lambda^\xi. \quad (2.11)$$

The last relations, following from the dimensionality considerations, (more precisely, see the next section) define the typical UV momentum scale  $\Lambda$ . By rescaling the fields, the

**Table 1.** Canonical dimensions of the fields and parameters in the model (2.9).

$F$	$\psi, \psi^\dagger$	$\mathbf{v}$	$\lambda, \lambda_0$	$\tau, \tau_0$	$m, \mu, \Lambda$	$D_0$	$u_0 = g_0^2$	$w_0$	$g, u, w$
$d_F^k$	$d/2$	$-1$	$-2$	$2$	$1$	$-2 + \xi$	$4 - d$	$\xi$	$0$
$d_F^\omega$	$0$	$1$	$1$	$0$	$0$	$1$	$0$	$0$	$0$
$d_F$	$d/2$	$1$	$0$	$2$	$1$	$\xi$	$4 - d$	$\xi$	$0$

constant  $w_0$  can be placed in front of the interaction term  $-\psi^\dagger(v\partial)\psi$ , which is more familiar for the field theory. We do not do it, however, in order not to spoil the natural form of the covariant derivative, and thus assign the factor  $w_0$  to the propagator  $\langle vv \rangle_0$ .

### 3. Canonical dimensions, UV divergences and the renormalization

It is well known that the analysis of UV divergences is based on the analysis of canonical dimensions (“power counting”); see e.g. [11, 12]. Dynamic models of the type (2.9), in contrast to static ones, have two independent scales: the time scale  $T$  and the length scale  $L$ . Thus the canonical dimension of some quantity  $F$  (a field or a parameter in the action functional) is completely characterized by two numbers, the frequency dimension  $d_F^\omega$  and the momentum dimension  $d_F^k$ , defined such that  $[F] \sim [T]^{-d_F^\omega} [L]^{-d_F^k}$ . These dimensions are found from the obvious normalization conditions

$$d_k^k = -d_{\mathbf{x}}^k = 1, \quad d_k^\omega = d_{\mathbf{x}}^\omega = 0, \quad d_\omega^k = d_t^k = 0, \quad d_\omega^\omega = -d_t^\omega = 1,$$

and from the requirement that each term of the action functional be dimensionless (with respect to the momentum and frequency dimensions separately). Then, based on  $d_F^k$  and  $d_F^\omega$ , one can introduce the total canonical dimension  $d_F = d_F^k + 2d_F^\omega$  (in the free theory,  $\partial_t \propto \partial^2$ ), which plays in the theory of renormalization of dynamical models the same role as the conventional (momentum) dimension does in static problems; see Chap. 5 of [12].

The dimensions for the model (2.9) are given in table 1, including renormalized parameters (without subscript “o”), which will be introduced later on. From table 1 or, equivalently, from the relations (2.11) it follows that the model is logarithmic (the both coupling constants  $g_0$  and  $w_0$  are simultaneously dimensionless) at  $d = 4$  and  $\xi = 0$ . Thus the UV divergences in the Green functions manifest themselves as poles in  $\varepsilon = 4 - d$ ,  $\xi$  and, in general, their linear combinations.

The total canonical dimension of an arbitrary 1-irreducible Green function  $\Gamma = \langle \Phi \cdots \Phi \rangle_{1-\text{ir}}$  is given by the relation [12]

$$d_\Gamma = d_\Gamma^k + 2d_\Gamma^\omega = d + 2 - N_\Phi d_\Phi, \quad (3.1)$$

where  $N_\Phi = \{N_\psi, N_{\psi^\dagger}, N_v\}$  are the numbers of corresponding fields entering into the function  $\Gamma$ , and the summation over all types of the fields is implied.

The total dimension  $d_\Gamma$  in logarithmic theory (that is, at  $\varepsilon = \xi = 0$ ) is the formal index of the UV divergence  $\delta_\Gamma = d_\Gamma|_{\varepsilon=\xi=0}$ . Superficial UV divergences, whose removal requires counterterms, can be present only in those functions  $\Gamma$  for which  $\delta_\Gamma$  is a non-negative integer. From table 1 and (3.1) we find

$$\delta_\Gamma = 6 - 2N_\psi - 2N_{\psi^\dagger} - N_v. \quad (3.2)$$

We recall that in our model nonvanishing Green functions must involve the both fields  $\psi$  and  $\psi^\dagger$  (see the discussion in the preceding section), so in (3.2) it is sufficient to take  $N_\psi \geq 1$  and (simultaneously)  $N_{\psi^\dagger} \geq 1$ . Straightforward analysis of the expression (3.2) then shows that superficial UV divergences can be present only in the following 1-irreducible functions:

$$\begin{aligned} \langle \psi^\dagger \psi \rangle \quad (\delta = 2) \quad & \text{with the counterterms} \quad \psi^\dagger \partial_t \psi, \quad \psi^\dagger \partial^2 \psi, \quad \psi^\dagger \psi, \\ \langle \psi^\dagger \psi \psi \rangle \quad (\delta = 0) \quad & \text{with the counterterm} \quad \psi^\dagger \psi^2, \\ \langle \psi^\dagger \psi^\dagger \psi \rangle \quad (\delta = 0) \quad & \text{with the counterterm} \quad (\psi^\dagger)^2 \psi, \\ \langle \psi^\dagger \psi v \rangle \quad (\delta = 1), \end{aligned}$$

for which the counterterm necessarily reduces to the form  $\psi^\dagger(v\partial)\psi = -\psi(v\partial)\psi^\dagger$  owing to the transversality of the velocity field. All such terms are present in the action (2.9), so that our model appears multiplicatively renormalizable.

The superficial divergence in the function  $\langle \psi^\dagger \psi v v \rangle$  with  $\delta = 0$  and the counterterm  $\psi^\dagger \psi v^2$ , allowed by the dimension, is in fact forbidden by the Galilean symmetry. Furthermore, the latter requires that the counterterms  $\psi^\dagger \partial_t \psi$  and  $\psi^\dagger(v\partial)\psi$  enter the renormalized action only in the form of the Lagrangian derivative  $\psi^\dagger \nabla_t \psi$ .

Strictly speaking, the arguments based on the Galilean symmetry are applicable only to the velocity field governed by the Navier–Stokes equation, and generally become invalid for synthetic Gaussian velocity ensembles. It turns out, however, that for a Gaussian ensemble of the type (2.7) with *vanishing* correlation time the Galilean symmetry of the counterterms indeed takes place; see e.g. [27]. This issue, along with the consequences of the Galilean invariance for the renormalization, is discussed in the appendix A of Ref. [21] in detail. The proof given there is fully applicable to the model (2.9). From the symmetry (2.4) it also follows that the trilinear counterterms enter the renormalized action as the single combination  $(\psi^\dagger)^2 \psi - \psi^\dagger \psi^2$ .

We thus conclude that the renormalized action can be written in the form

$$\begin{aligned} \mathcal{S}_R(\Phi) = & \psi^\dagger \left( -Z_1 \nabla_t + Z_2 \lambda \partial^2 - Z_3 \lambda \tau \right) \psi + \\ & + Z_4 \frac{\lambda g}{2} \left( (\psi^\dagger)^2 \psi - \psi^\dagger \psi^2 \right) + \mathcal{S}(\mathbf{v}). \end{aligned} \quad (3.3)$$

Here  $\lambda$ ,  $\tau$  and  $g$  are renormalized analogs of the bare parameters (with the subscripts “o”) and  $\mu$  is the reference mass scale (additional arbitrary parameter of the renormalized theory). Since the last term  $\mathcal{S}(\mathbf{v})$  given by (2.10) is not renormalized, the amplitude  $D_0$  is expressed in renormalized parameters as

$$D_0 = w_0 \lambda_0 = w \lambda \mu^\xi. \quad (3.4)$$



Expression (3.3) can be obtained by the multiplicative renormalization of the fields  $\psi \rightarrow \psi Z_\psi$ ,  $\psi^\dagger \rightarrow \psi^\dagger Z_{\psi^\dagger}$ ,  $v \rightarrow v Z_v$  and the parameters:

$$\lambda_0 = \lambda Z_\lambda, \quad \tau_0 = \tau Z_\tau, \quad g_0 = g \mu^{\varepsilon/2} Z_g, \quad w_0 = w \mu^\xi Z_w. \quad (3.5)$$

The renormalization constants in Eqs. (3.3) and (3.5) are related as follows:

$$\begin{aligned} Z_1 &= Z_\psi Z_{\psi^\dagger} = Z_v Z_\psi Z_{\psi^\dagger} & Z_2 &= Z_\psi Z_{\psi^\dagger} Z_\lambda, & Z_3 &= Z_\psi Z_{\psi^\dagger} Z_\lambda Z_\tau, \\ Z_4 &= Z_\psi Z_{\psi^\dagger}^2 Z_g Z_\lambda = Z_\psi^2 Z_{\psi^\dagger} Z_g Z_\lambda, & 1 &= Z_w Z_\lambda. \end{aligned} \quad (3.6)$$

Resolving these relations with respect to the renormalization constants of the fields and parameters gives

$$\begin{aligned} Z_v &= 1, & Z_\psi &= Z_{\psi^\dagger} = Z_1^{1/2}, & Z_\tau &= Z_3 Z_2^{-1}, \\ Z_\lambda &= Z_w^{-1} = Z_2 Z_1^{-1}, & Z_g &= Z_4 Z_2^{-1} Z_1^{-1/2}, \end{aligned} \quad (3.7)$$

where the first equality is a consequence of the Galilean symmetry and the second – a consequence of the symmetry (2.4). The first relation in the second line is a consequence of the absence of renormalization of the term  $\mathcal{S}(\mathbf{v})$  in (3.3). For the coupling constant  $u_0 = g_0^2$  introduced in (2.11) one has:

$$u_0 = u \mu^\varepsilon Z_u, \quad Z_u = Z_g^2. \quad (3.8)$$

The renormalization constants  $Z_1$ – $Z_4$  are calculated directly from the diagrams, then the constants in (3.5) are found from (3.7) and (3.8).

The renormalization constants capture all the divergences at  $\varepsilon, \xi \rightarrow 0$ , so that the correlation functions of the renormalized model (3.3) have finite limits for  $\varepsilon, \xi = 0$  when expressed in renormalized parameters  $\lambda, \tau$  and so on. In practical calculations, we used the minimal subtraction (MS) scheme, in which the renormalization constants have the forms  $Z_i = 1 +$  only singularities in  $\varepsilon$  and  $\xi$ , with the coefficients depending on the two completely dimensionless parameters — renormalized coupling constants  $u$  and  $w$ . To simplify the resulting expressions, it is convenient to pass to the new couplings,

$$u \rightarrow u/16\pi^2, \quad w \rightarrow w/16\pi^2; \quad (3.9)$$

in what follows they will be denoted by the same symbols.

Explicit calculation in the first order in  $u$  and  $w$  (one-loop approximation) gives rather simple results:

$$Z_1 = 1 + \frac{u}{4\varepsilon}, \quad Z_2 = 1 + \frac{u}{8\varepsilon} - \frac{3w}{4\xi}, \quad Z_3 = 1 + \frac{u}{2\varepsilon}, \quad Z_4 = 1 + \frac{u}{\varepsilon}, \quad (3.10)$$

with the corrections of second order in  $u$  and  $w$  and higher. For  $w = 0$  one obtains (up to the notation) the well-known one-loop result for the model (2.3); cf. e.g. [3]), while for  $u = 0$  the exact result for the rapid-change model is recovered; cf. e.g. [28].

#### 4. RG functions and RG equations

Let us recall an elementary derivation of the RG equations; detailed exposition can be found in monographs [11, 12]. The RG equations are written for the renormalized

Green functions  $G_R = \langle \Phi \cdots \Phi \rangle_R$ , which differ from the original (unrenormalized) ones  $G = \langle \Phi \cdots \Phi \rangle$  only by normalization (due to rescaling of the fields) and choice of parameters, and therefore can equally be used for analyzing the critical behaviour. The relation  $\mathcal{S}_R(\Phi, e, \mu) = \mathcal{S}(\Phi, e_0)$  between the functionals (2.9) and (3.3) results in the relations

$$G(e_0, \dots) = Z_\psi^{N_\psi} Z_{\psi^\dagger}^{N_{\psi^\dagger}} G_R(e, \mu, \dots). \quad (4.1)$$

between the Green functions. Here, as usual,  $N_\psi$  and  $N_{\psi^\dagger}$  are the numbers of corresponding fields entering into  $G$  (we recall that in our model  $Z_v = 1$ );  $e_0 = \{\lambda_0, \tau_0, u_0, w_0\}$  is the full set of bare parameters and  $e = \{\lambda, \tau, u, w\}$  are their renormalized counterparts; the dots stand for the other arguments (times/frequencies and coordinates/momenta).

We use  $\tilde{\mathcal{D}}_\mu$  to denote the differential operation  $\mu \partial_\mu$  for fixed  $e_0$  and operate on both sides of the equation (4.1) with it. This gives the basic RG differential equation:

$$\{\mathcal{D}_{RG} + N_\psi \gamma_\psi + N_{\psi^\dagger} \gamma_{\psi^\dagger}\} G_R(e, \mu, \dots) = 0, \quad (4.2)$$

where  $\mathcal{D}_{RG}$  is the operation  $\tilde{\mathcal{D}}_\mu$  expressed in the renormalized variables:

$$\mathcal{D}_{RG} \equiv \mathcal{D}_\mu + \beta_u \partial_u + \beta_w \partial_w - \gamma_\lambda \mathcal{D}_\lambda - \gamma_\tau \mathcal{D}_\tau. \quad (4.3)$$

Here we have written  $\mathcal{D}_x \equiv x \partial_x$  for any variable  $x$ , and the anomalous dimensions  $\gamma$  are defined as

$$\gamma_F \equiv \tilde{\mathcal{D}}_\mu \ln Z_F \quad \text{for any quantity } F, \quad (4.4)$$

and the  $\beta$  functions for the two dimensionless couplings  $u$  and  $w$  are

$$\beta_u \equiv \tilde{\mathcal{D}}_\mu u = u [-\varepsilon - \gamma_u], \quad \beta_w \equiv \tilde{\mathcal{D}}_\mu w = w [-\xi - \gamma_w], \quad (4.5)$$

where the second equalities come from the definitions and the relations (3.5).

Equations (3.7) result in the following relations between the anomalous dimensions (4.4):

$$\begin{aligned} \gamma_\psi = \gamma_{\psi^\dagger} = \gamma_1/2, \quad \gamma_\lambda = -\gamma_w = \gamma_2 - \gamma_1, \quad \gamma_v = 0, \\ \gamma_\tau = \gamma_3 - \gamma_2, \quad \gamma_u = 2\gamma_4 - 2\gamma_2 - \gamma_1. \end{aligned} \quad (4.6)$$

The anomalous dimension corresponding to a given renormalization constant  $Z_F$  is readily found from the relation

$$\gamma_F = (\beta_u \partial_u + \beta_w \partial_w) \ln Z_F \simeq -(\varepsilon \mathcal{D}_u + \xi \mathcal{D}_w) \ln Z_F. \quad (4.7)$$

In the first relation, we used the definition (4.4), expression (4.3) for the operation  $\tilde{\mathcal{D}}_\mu$  in renormalized variables, and the fact that the  $Z$ 's depend only on the two completely dimensionless coupling constants  $u$  and  $w$ . In the second (approximate) relation, we retained only the leading-order terms in the  $\beta$  functions (4.5), which is sufficient for the first-order approximation. The factors  $\varepsilon$  and  $\xi$  in (4.7) cancel the corresponding poles contained in the expressions (3.10) for the constants  $Z_F$ , which leads to the final UV finite expressions for the anomalous dimensions. This gives:

$$\gamma_1 = -\frac{u}{4}, \quad \gamma_2 = -\frac{u}{8} + \frac{3w}{4}, \quad \gamma_3 = -\frac{u}{2}, \quad \gamma_4 = -u, \quad (4.8)$$

and from (4.6) one finally obtains:

$$\begin{aligned}\gamma_\psi = \gamma_{\psi^\dagger} &= -\frac{u}{8}, & \gamma_\lambda = -\gamma_w &= \frac{u}{8} + \frac{3w}{4}, \\ \gamma_\tau &= -\frac{3u}{8} - \frac{3w}{4}, & \gamma_u &= -\frac{3u}{2} - \frac{3w}{2},\end{aligned}\tag{4.9}$$

with corrections of order  $u^2$ ,  $w^2$ ,  $uw$  and higher.

## 5. Fixed points and scaling regimes

It is well known that the long-time large-distance asymptotic behaviour of a renormalizable field theory is determined by the IR attractive fixed points of the corresponding RG equations. In general, coordinates of the possible fixed points are found from the requirement that all the  $\beta$  functions vanish. In the model (2.9) the coordinates  $u_*$ ,  $w_*$  are determined by the two equations

$$\beta_u(u_*, w_*) = 0, \quad \beta_w(u_*, w_*) = 0,\tag{5.1}$$

with the  $\beta$  functions given in (4.5). The type of a fixed point is determined by the matrix

$$\Omega = \{\Omega_{ij} = \partial\beta_i/\partial g_j\},\tag{5.2}$$

where  $\beta_i$  is the full set of the  $\beta$  functions and  $g_j = \{g, w\}$  is the full set of coupling constants. For an IR attractive fixed point the matrix  $\Omega$  is positive, that is, the real parts of all its eigenvalues are positive.

From the definitions (4.5) and explicit expressions (4.9) for the anomalous dimensions we derive the following leading-order expressions for the  $\beta$  functions:

$$\beta_u = u(-\varepsilon + 3u/2 + 3w/2), \quad \beta_w = w(-\xi + u/8 + 3w/4).\tag{5.3}$$

From Eqs. (5.1) and (5.3) we can identify four different fixed points. For the three of them, the matrix  $\Omega$  appears triangular and its eigenvalues are given by the diagonal elements  $\Omega_u = \partial\beta_u/\partial u$  and  $\Omega_w = \partial\beta_w/\partial w$ .

1. Gaussian (free) fixed point:  $u_* = w_* = 0$ ;  $\Omega_u = -\varepsilon$ ,  $\Omega_w = -\xi$  (all these expressions are exact).

2.  $w_* = 0$  (exact result to all orders),  $u_* = 2\varepsilon/3$ ;  $\Omega_u = \varepsilon$ ,  $\Omega_w = -\xi + \varepsilon/12$ .

In this regime, effects of the turbulent mixing are irrelevant in the leading-order IR asymptotic behaviour; the basic critical exponents are independent on  $\xi$  and coincide to all orders with their counterparts for the “pure” DP class [1]–[3]. However, the dependence on  $\xi$  appears in the *corrections* to the leading-order behaviour, in particular, due to the correction exponent  $\Omega_w$ . Although the expression for  $\Omega_u$  is not exact (it has corrections of order  $\varepsilon^2$  and higher), the inequality  $\Omega_u > 0$  is equivalent to  $\varepsilon > 0$  within the  $\varepsilon$  expansion.

3.  $u_* = 0$ ,  $w_* = 4\xi/3$  (exact);  $\Omega_u = -\varepsilon + 2\xi$ ,  $\Omega_w = \xi$  (exact).

In this regime, the nonlinearity  $(\psi^\dagger)^2\psi - \psi^\dagger\psi^2$  of the DP model is irrelevant, and we arrive at the rapid-change model of a passively advected scalar field  $\psi$  [27]. For that model, the  $\beta$  function is given exactly by the one-loop approximation (see [28]), hence the exact results for  $w_*$  and  $\Omega_w$ . The dependence on  $\varepsilon$  appears in the *corrections*, in particular, due to the correction exponent  $\Omega_u$ .

4.  $u_* = 4(\varepsilon - 2\xi)/5$ ,  $w_* = 2(12\xi - \varepsilon)/15$ . The eigenvalues of the matrix (5.2) have the forms:

$$\lambda^\pm = \frac{1}{20} \left( 11\varepsilon - 12\xi \pm \sqrt{161\varepsilon^2 - 824\varepsilon\xi + 1104\xi^2} \right). \quad (5.4)$$

It is easily checked that they are both real for all  $\varepsilon$  and  $\xi$  (the expression in the square root is positive definite) and positive for  $\varepsilon/12 < \xi < \varepsilon/2$ .

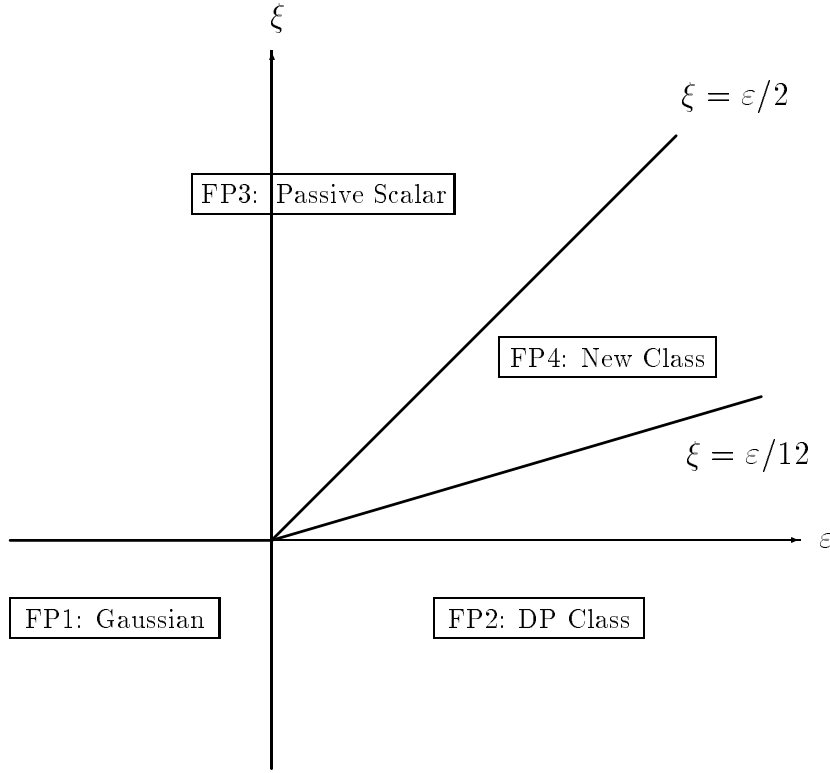
This fixed point corresponds to a new nontrivial IR scaling regime (universality class), in which the nonlinearity of the DP model (2.3) and the turbulent mixing are simultaneously important; the corresponding critical exponents depend on the both RG expansion parameters  $\varepsilon$  and  $\xi$  and are calculated as double series in these parameters; see section 6.

In figure 1 we show the regions of IR stability for all the fixed points in the  $\varepsilon$ - $\xi$  plane, that is, the regions in which the eigenvalues of the matrix (5.2) for the given fixed point are both positive.

In the one-loop approximation (5.3), all the boundaries of the regions of stability are given by straight rays; there are neither gaps nor overlaps between the different regions. <sup>‡</sup> Such pattern is typical of the first-order approximations; cf. [7, 10, 20, 21, 29] for various models. The boundaries  $\varepsilon < 0$ ,  $\xi < 0$  for point 1,  $\varepsilon > 0$  for point 2 and  $\xi > 0$  for point 3 are exact, while the other can be affected by the higher-order corrections: the boundaries will become curved and gaps or overlaps can appear between the different regions of IR stability.

It is important that, for all these fixed points, the coordinates  $u_*$ ,  $w_*$  are non-negative in the regions of their IR stability, in agreement with the physical meaning of these parameters. It is also worth noting that the both boundaries for point 4 are determined by the same eigenvalue  $\lambda^-$ , which changes its sign at  $\xi = \varepsilon/2$  and  $\xi = \varepsilon/12$ , while  $\lambda^+$  remains strictly positive in the entire region of stability.

<sup>‡</sup> For the first three fixed point this is obvious from the expressions for  $\Omega_{u,w}$ ; for the point 4 this is quite unexpected at the first sight, but can be explained by the homogeneity of the expressions (5.4) in  $\varepsilon$  and  $\xi$ , and is easily seen from the simple form of the determinant,  $\det \Omega = (\varepsilon - 2\xi)(12\xi - \varepsilon)/10$ .



**Figure 1.** Regions of stability of the fixed points in the model (2.9).

## 6. Critical scaling and critical dimensions

Existence of an IR attractive fixed point implies existence of scaling (self-similar) behaviour of the Green functions in the IR range. In this “critical scaling” all the “IR irrelevant” parameters ( $\lambda$ ,  $\mu$ ,  $g$  and  $w$  in our case) are fixed and the “IR relevant” parameters (coordinates/momenta, times/frequencies,  $\tau$  and the fields) are dilated. The critical dimensions  $\Delta_F$  of the IR relevant quantities  $F$  are given by the relations

$$\Delta_F = d_F^k + \Delta_\omega d_F^\omega + \gamma_F^*, \quad \Delta_\omega = 2 - \gamma_\lambda^*, \quad (6.1)$$

with the normalization condition  $\Delta_k = 1$ ; see e.g. [12] for more detail. Here  $d_F^{k,\omega}$  are the canonical dimensions of  $F$ , given in table 1, and  $\gamma_F^*$  is the value of the anomalous dimension (4.4) at the fixed point:  $\gamma_F^* = \gamma_F(u_*, w_*)$ . This gives

$$\Delta_\psi = \Delta_{\psi^\dagger} = d/2 + \gamma_\psi^*, \quad \Delta_\tau = 2 + \gamma_\tau^*. \quad (6.2)$$

Four fixed points of the model (2.3) revealed in the preceding section correspond to four possible IR scaling regimes; for given  $\varepsilon$  and  $\xi$  only one of them is IR attractive and governs the IR behaviour. From the general expressions (6.1), (6.2) and the explicit one-loop expressions (4.9) we find:

1. Gaussian (free) fixed point; all the expressions are exact:

$$\Delta_\psi = d/2, \quad \Delta_\tau = \Delta_\omega = 2. \quad (6.3)$$

2. Directed percolation (DP) regime; mixing irrelevant:

$$\Delta_\psi = 2 - 7\varepsilon/12, \quad \Delta_\tau = 2 - \varepsilon/4, \quad \Delta_\omega = 2 - \varepsilon/12. \quad (6.4)$$

These dimensions depend only on  $\varepsilon$ , the corrections of order  $O(\varepsilon^2)$  and the references can be found in [3].  $\mathcal{S}$  The  $O(\varepsilon^3)$  calculation is in progress; see [30].

3. Obukhov–Kraichnan exactly soluble regime; all results exact (see e.g. [28] for the detailed discussion):

$$\Delta_\omega = \Delta_\tau = 2 - \xi, \quad \Delta_\psi = d/2. \quad (6.5)$$

4. New universality class (both mixing and DP interaction are relevant):

$$\Delta_\psi = 2 + (\xi - 3\varepsilon)/5, \quad \Delta_\tau = 2 - (\varepsilon + 3\xi)/5, \quad \Delta_\omega = 2 - \xi \text{ (exact)}. \quad (6.6)$$

The first two dimensions have nontrivial higher-order corrections in  $\varepsilon$  and  $\xi$ . The exact results for  $\Delta_\omega$  in (6.5) and (6.6) follow from the general relations  $\gamma_\lambda = \gamma_w$  in (4.6) and  $\Delta_\omega = 2 - \gamma_\lambda^*$  in (6.1) and the identity  $\gamma_w = \xi$ , which is a consequence of the fixed-point equation  $\beta_w = 0$  with  $\beta_w$  from (4.5) for any fixed point with  $w_* \neq 0$ .

Let us illustrate the consequences of these general scaling relations for the spreading of a cloud of the agent (or a cloud of “infected” particles) in the turbulent environment. The mean-square radius  $R(t)$  at time  $t > 0$  of a cloud of such particles, which started from the origin  $\mathbf{x}' = 0$  at time  $t' = 0$ , is related with the linear response function in the time-coordinate representation as follows:

$$R^2(t) = \int d\mathbf{x} \, x^2 G(t, \mathbf{x}), \quad G(t, \mathbf{x}) = \langle \psi(t, \mathbf{x}) \psi^\dagger(0, \mathbf{0}) \rangle, \quad x = |\mathbf{x}|, \quad (6.7)$$

see e.g. [31]. For the response function, the relations (6.3)–(6.6) result in the following IR asymptotic expression:

$$G(t, \mathbf{x}) = x^{-2\Delta_\psi} F\left(\frac{x}{t^{1/\Delta_\omega}}, \frac{\tau}{t^{\Delta_\tau/\Delta_\omega}}\right), \quad (6.8)$$

with some scaling function  $F$ . Substituting (6.8) into (6.7) gives (with the assumption that the integral converges) the desired scaling expression for the radius:

$$R^2(t) = t^{(d+2-2\Delta_\psi)/\Delta_\omega} f\left(\frac{\tau}{t^{\Delta_\tau/\Delta_\omega}}\right) \quad (6.9)$$

where the scaling function  $f$  is related with  $F$  from (6.8) as follows:

$$f(z) = \int d\mathbf{x} \, x^{2-2\Delta_\psi} F(x, z).$$

Directly at the critical point (assuming that the function  $f$  is finite at  $\tau = 0$ ) one obtains from (6.9) the power law for the radius:

$$R^2(t) \propto t^{(d+2-2\Delta_\psi)/\Delta_\omega} = t^{(2-2\gamma_\psi^*)/(2-\gamma_\lambda^*)}, \quad (6.10)$$

$\mathcal{S}$  The conventional critical exponents used e.g. in [3] are related to the critical dimensions from (6.4) as  $z = \Delta_\omega$ ,  $1/\nu = \Delta_\tau$ ,  $d + \eta = 2\Delta_\psi$ .

in the second equality we used the relations (6.1) and (6.2). For the Gaussian fixed point (6.3) the usual “1/2 law”  $R(t) \propto t^{1/2}$  for the ordinary random walk is recovered. For the fixed point (6.5), where the DP nonlinearity is irrelevant, one obtains the exact result  $R(t) \propto t^{1/(2-\xi)}$ . For the most realistic (Kolmogorov) value  $\xi = 4/3$  this gives  $R(t) \propto t^{3/2}$  in agreement with Richardson’s “4/3 law”  $dR^2/dt \propto R^{4/3}$  for a passively advected scalar impurity; see [31]. For the other two fixed points the exponents in (6.9), (6.10) are given by infinite series in  $\varepsilon$  (point 2) or  $\varepsilon$  and  $\xi$  (point 4); the first-order approximations are easily obtained from (6.4) and (6.6).

## 7. Conclusion

We studied effects of turbulent stirring and mixing on the reaction-diffusion system in which the spreading of an agent (e.g., infectious disease) occurs, as an example of critical behaviour in an nonequilibrium system near its transition between the absorbing and fluctuating phases. We coupled two paradigmatic models: the so-called simple epidemic process, also known as Gribov’s process and equivalent to Reggeon field theory (2.3), directed bond percolation process, Schlögl’s first reaction and so on [1]–[3], and the Obukhov–Kraichnan’s rapid-change model (2.8) for the advecting turbulent velocity field [27]. The full problem can be reformulated as a multiplicatively renormalizable field theoretic model (2.9), which allows one to apply the field theoretic renormalization group [11, 12] to the analysis of its critical behaviour.

We showed that, depending on the relation between the spatial dimension  $d$  and the exponent  $\xi$  that comes from the Obukhov–Kraichnan’s ensemble, the model exhibits four different critical regimes, associated with four fixed points of the RG equations. Three fixed points correspond to known regimes: Gaussian fixed point (ordinary diffusion or random walk), directed percolation process with no advection (DP class), and passively advected scalar field (reaction processes, described by Gribov’s nonlinearity, appear irrelevant). The fourth point reveals existence of a new nonequilibrium universality class, in which both the reaction and the turbulent mixing are relevant; the corresponding critical exponents are calculated to the leading order (one-loop approximation) of the double expansion in  $\xi$  and  $\varepsilon = 4 - d$ .

Judging naively from the dimensions of the coupling constants (2.11) one could expect that the latter regime must take place when  $\xi$  and  $\varepsilon$  are simultaneously positive, but the careful RG analysis shows that the region of IR stability of the corresponding fixed point is much narrower (in the one-loop level it shrinks to the sector  $\varepsilon/12 < \xi < \varepsilon/2$ ). Of course, this is not a big surprise: similar phenomenon was established a long ago for the interplay between the long-range and short-range spin-spin interactions in models of equilibrium critical behaviour [32]. Later it was also encountered in the long-range versions of various models with nonequilibrium transitions [7, 33] and in models of a passive scalar advected by the velocity field with finite correlation time [29].

In our case, this effect leads to interesting prediction: in contrary to what could be naively anticipated, the most realistic spatial dimensions  $d = 2$  or  $3$  and the

Kolmogorov's exponent  $\xi = 4/3$  for the fully developed turbulence correspond not to the most nontrivial new regime, but to the passive-scalar fixed point: the nonlinearity of the Reggeon model is irrelevant and the spreading of the agent is completely determined by the turbulent transfer. In particular, the time spreading of a cloud of infected particles (or of the agent) behaves accordingly to the power law  $R(t) \propto t^{1/(2-\xi)}$ , which is the proper generalization of Richardson's law to the case of arbitrary exponent  $\xi$  in the velocity correlator (2.8).

Further investigation should take into account anisotropy of the experimental set-up, compressibility, non-Gaussian character and finite correlation time of the advecting velocity field, effects of memory (immunization); interaction of the order parameter with other relevant degrees of freedom (mode-mode coupling), feedback of the reaction on the dynamics of the velocity and so on. This work is already in progress.

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## Appendix A. Calculation of the renormalization constants

In this section we derive the first-order results (3.10) for the renormalization constants. Of course, the one-loop calculation is rather simple and can be accomplished in a few different ways, but we will discuss it for completeness and in order to mention some interesting subtleties specific of the model (2.9).

The renormalization constants can be found from the requirement that the Green functions of the renormalized model (3.3), when expressed in renormalized variables, be UV finite (in our case, be finite at  $\varepsilon \rightarrow 0$ ,  $\xi \rightarrow 0$ ). Owing to the symmetry (2.4) and to the Galilean invariance, the full set of constants  $Z_1$ – $Z_4$  in our model can be found from just two 1-irreducible functions:  $\langle \psi^\dagger \psi \rangle_{1\text{-ir}}$  and  $\langle \psi^\dagger \psi^\dagger \psi \rangle_{1\text{-ir}}$  (or, if desired,  $\langle \psi^\dagger \psi \psi \rangle_{1\text{-ir}}$ ). In the renormalized model, the corresponding one-loop approximations have the forms

$$\langle \psi^\dagger \psi \rangle_{1\text{-ir}} = i\omega Z_1 + \lambda p^2 Z_2 + \lambda \tau Z_3 - \frac{1}{2} \rightarrow \text{bubble diagram} - \text{tadpole diagram} \quad (\text{A.1})$$

(which is the Dyson equation for the exact function  $\langle \psi^\dagger \psi \rangle$ , hence the minus signs



in front of the diagrams) and

$$\langle \psi^\dagger \psi^\dagger \psi \rangle_{1-\text{ir}} = gZ_4 + 2 \text{ (triangle with 3 solid lines)} + 2 \text{ (triangle with 2 solid, 1 wavy)} + \frac{1}{2} \text{ (triangle with 1 solid, 2 wavy)} \quad (\text{A.2})$$

The first two diagrams in (A.2) have in fact two different forms, related by the mirror reflection, but they give equal contributions to the renormalization constants and are accounted by the factors of 2. Owing to the Galilean symmetry, the constant  $Z_1$  can also be found from the function

$$\langle \psi^\dagger \psi \mathbf{v} \rangle_{1-\text{ir}} = -i\mathbf{p}Z_1 + \text{ (triangle with 2 solid, 1 wavy)} + \text{ (triangle with 1 solid, 2 wavy)} \quad (\text{A.3})$$

The solid lines in (A.1)–(A.3) denote the propagator  $\langle \psi \psi^\dagger \rangle_0$  from (2.5) (the arrow is directed from  $\psi$  to  $\psi^\dagger$ ) and the wavy lines correspond to  $\langle vv \rangle_0$  from (2.8). All the diagrammatic elements should be expressed in renormalized variables using the relations (3.3)–(3.6). In the one-loop approximation, the  $Z$ 's in the bare terms of (A.1), (A.2) should be taken in the first order in  $u = g^2$  and  $w$ , while in the diagrams they should simply be replaced with unities,  $Z_i \rightarrow 1$ . Thus the passage to renormalized variables in the diagrams is achieved by the simple substitutions  $\lambda_0 \rightarrow \lambda$ ,  $\tau_0 \rightarrow \tau$ ,  $g_0 \rightarrow g\mu^{\varepsilon/2}$  and  $w_0 \rightarrow w\mu^\xi$ .

The second and third diagrams in (A.2) and the second diagram in (A.3) appear UV finite and therefore give no contribution to the renormalization constants. Indeed, due to the transversality of the velocity field, the derivative in the vertex  $-\psi^\dagger(v\partial)\psi$  can also be moved onto the field  $\psi^\dagger$  using integration by parts:  $-\psi^\dagger(v\partial)\psi = \psi(v\partial)\psi^\dagger$ . Thus in any diagram involving  $n$  external vertices of this type, the factor  $p^n$  with  $n$  external momenta  $p$  will be taken outside the corresponding integrals. This reduces the dimension of the integrand by  $n$  units and can make it UV convergent. In the case at hand, this proves the UV finiteness of the three diagrams mentioned above: for all of them  $n = 2$  while the formal index of divergence is 0 or 1.

What is more, since the propagator (2.5) is retarded and (2.8) contains the  $\delta$  function in time, the second diagrams both in (A.2) and (A.3) contain self-contracted “circuits” of the step functions in time and therefore vanish identically. This argument, however, does not apply to the second diagram in (A.1), which requires a more careful treatment; cf. [21, 28]. The analytic expression for that diagram has the form

$$-p_i p_j \int \frac{d\omega}{(2\pi)} \int_{k>m} \frac{d\mathbf{k}}{(2\pi)^d} \frac{D_0 P_{ij}(\mathbf{k})}{k^{d+\xi}} \frac{1}{-i\omega + \sigma(\mathbf{p} - \mathbf{k})}, \quad (\text{A.4})$$

where the prefactor comes from the vertices, the IR cutoff and the first cofactor in the integrand come from the propagator (2.8) and the second cofactor with  $\sigma(\mathbf{k}) = \lambda(k^2 + \tau)$  is the propagator (2.5). The expression (A.4) is independent of the external frequency. Integration over  $\omega$  involves the indeterminacy

$$\int \frac{d\omega}{(2\pi)} \frac{1}{-i\omega + \sigma(\mathbf{p} - \mathbf{k})} = \theta(0), \quad (\text{A.5})$$

where  $\theta(0)$  is the step function at the origin. This reflects the details of the velocity statistics lost in the white-noise limit; see the discussion in [27]. In the case at hand, the  $\delta$  function in (2.7) should be understood as the limit of a narrow function which is necessarily *symmetric* in  $t, t'$ , because one deals with a pair correlator. Thus the indeterminacy in (A.5) must be unambiguously resolved as half the sum of the limits:  $\theta(0) = 1/2$ . Then the remaining integral over  $\mathbf{k}$  in (A.4) appears independent of  $\mathbf{p}$  and  $\tau$  and is easily calculated:

$$\int_{k>m} \frac{d\mathbf{k}}{(2\pi)^d} \frac{P_{ij}(\mathbf{k})}{k^{d+\xi}} = \frac{\delta_{ij}(d-1)}{d} \int_{k>m} \frac{d\mathbf{k}}{(2\pi)^d} \frac{1}{k^{d+\xi}} = \delta_{ij} m^{-\xi} \frac{S_d}{(2\pi)^d} \frac{(d-1)}{d\xi},$$

where  $S_d = 2\pi^{d/2}/\Gamma(d/2)$  with Euler's  $\Gamma$  function is the surface area of the unit sphere in the  $d$ -dimensional space. Collecting all the factors and setting  $D_0 = \lambda_0 w_0 = \lambda w \mu^\xi$  gives the final result for the diagram:

$$-\lambda w p^2 (\mu/m)^\xi \frac{(d-1)}{2d\xi} \frac{S_d}{(2\pi)^d} = -\lambda p^2 \frac{w}{16\pi^2} \frac{3}{4\xi} + \text{UV finite part.} \quad (\text{A.6})$$

In the last equality in (A.6), the UV divergent part of the diagram is selected (the replacements  $(\mu/m)^\xi \rightarrow 1$  and  $d = 4 - \varepsilon \rightarrow 4$  are made); it contains a first-order pole in  $\xi$ . The expression (A.6) as a whole is proportional to  $p^2$ , so that it gives a contribution only to the renormalization constant  $Z_2$  in (A.1).

The remaining two diagrams in (A.1), (A.2) do not involve the velocity correlator; they are independent of  $\xi$  and contain only poles in  $\varepsilon$ . Although the calculation of these diagrams is discussed in [3] within the context of the model (2.3), we will sketch an alternative calculation here, mainly in order to present a reference formula, which can be interesting in itself.

The key point is as follows: the convolution of two functions of the form

$$F(\alpha, a, \tau) \equiv (-i\omega a + k^2 + \tau)^{-\alpha}, \quad \tau > 0 \quad (\text{A.7})$$

is a function of the same form:

$$F(\alpha_1, a_1, \tau_1) * F(\alpha_2, a_2, \tau_2) = K(\alpha_1, \alpha_2; a_1, a_2) F(\alpha_3, a_3, \tau_3) \quad (\text{A.8})$$

if  $a_1$  and  $a_2$  have the same sign and zero otherwise. Here

$$a_3 = a_1 + a_2, \quad \alpha_3 = \alpha_1 + \alpha_2 - d/2 - 1, \quad \tau_3 = a_3(\tau_1/a_1 + \tau_2/a_2); \quad (\text{A.9})$$

the coefficient is independent on the  $\tau_{1,2}$  and has the form:

$$K(\alpha_1, \alpha_2; a_1, a_2) = a_1^{d/2-\alpha_1} a_2^{d/2-\alpha_2} a_3^{\alpha_1+\alpha_2-d-1} \frac{\Gamma(\alpha_3)}{(4\pi)^{d/2} \Gamma(\alpha_1) \Gamma(\alpha_2)}.$$

For the “massless” case  $\tau_{1,2} = 0$  this formula was proposed and used in the three-loop calculation of the critical exponent  $z$  in the so-called model A of critical dynamics in [34]; see also [21, 35] for other applications. For the general case ( $\tau_{1,2} \geq 0$ ) equation (A.8) can be obtained from the following observation: In the time-space representation the function (A.7) takes on the form

$$F(\alpha, a, \tau) \rightarrow \frac{\theta(t \text{ sign}(a)) a^{d/2-\alpha}}{(4\pi)^{d/2} \Gamma(\alpha)} t^{\alpha-d/2-1} \exp \left\{ -\frac{ax^2}{4t} - \tau t/a \right\}, \quad (\text{A.10})$$

and the product of such functions (which corresponds to the convolution of their Fourier transforms) is obviously a function of the same form. Note that, when  $a_1$  and  $a_2$  have different signs, the convolution in (A.8) corresponds to the product of a retarded and an advanced functions of the form (A.10) and therefore vanishes.

Let us turn to the remaining diagrams in (A.1), (A.2). With no loss of generality, one can set  $\lambda = 1$  (the dependence on  $\lambda$  can be restored in the final answers using the dimensionality considerations). Then the frequency-momentum integral that corresponds to the first diagram in (A.1), is equal to the convolution (A.8), in which  $a_{1,2} = 1$ ,  $\alpha_{1,2} = 1$ ,  $\tau_{1,2} = \tau$ . From (A.9) one obtains  $a_3 = 2$ ,  $\alpha_3 = -1 + \varepsilon/2$ ,  $\tau_3 = 4\tau$ , so that the left-hand side of (A.8) is proportional to

$$\Gamma(-1 + \varepsilon/2)(-2i\omega + k^2 + 4\tau)^{1-\varepsilon/2} = \frac{2}{\varepsilon}(-2i\omega + k^2 + 4\tau) + \text{UV finite part} \quad (\text{A.11})$$

(here and below we omit uninteresting finite numerical factors, powers of  $\pi$  and so on).

The first diagram in (A.2) is logarithmically divergent. According to the general statements of the renormalization theory, its divergent part in the MS scheme does not depend on the specific choice of the external momenta, frequencies, and “masses” like  $\tau$ , provided this choice guarantees IR convergence of the corresponding integral; see e.g. [11, 12]. Thus we can set the external frequency and momentum flowing into the right lower vertex equal to zero. (We could also set  $\tau = 0$ , but this is not necessary). Then the integral that to the diagram in the frequency-momentum representation, becomes equal to the convolution (A.8) with  $a_{1,2} = 1$ ,  $\alpha_1 = 1$ ,  $\alpha_2 = 2$ ,  $\tau_{1,2} = \tau$ . From (A.9) it follows  $a_3 = 2$ ,  $\alpha_3 = \varepsilon/2$ ,  $\tau_3 = 4\tau$ , and the left-hand side of (A.8) is proportional to

$$\Gamma(\varepsilon/2)(-2i\omega + k^2 + 4\tau)^{-\varepsilon/2} = \frac{2}{\varepsilon} + \text{UV finite part}. \quad (\text{A.12})$$

As expected, the expressions (A.11) and (A.12) contain first-order poles in  $\varepsilon$ . Their pole parts are polynomials in frequencies, momenta, and “masses,” so that they can be cancelled in expressions (A.1), (A.2) by the proper choice of the renormalization constants  $Z_1$ – $Z_4$ . Taking into account all the factors (signs, symmetry coefficients, factors  $\pm g$  from the vertices) and the replacement (3.9) gives the results announced in (3.10). It remains to note that the calculation of the 1-irreducible function (A.3) indeed gives the same result (3.10) for the constant  $Z_1$ , in agreement with general consequences of the Galilean invariance of our model.

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